Notes on Representation Theory

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About This Notes

Note. This is under construction!

This notes mostly amount to an amalgamation of thoughts and ideas I came across when studding the representation theory of groups. Many of the comments in here don't have much *mathematical* value per say – they are better understood as *philosophical* considerations on the topic at hand. The focus of this notes is the representation theory of Lie groups and algebras, but we'll dive into some other topics too. We'll assume basic knowledge of abstract algebra, group theory and differential geometry.

Lengthy proves are favored as opposed to collections of smaller lemmas, since we want to emphasize the relevant results.

Contents

Chapter 1

Groups & Actions

> A group is a groupoid with a single element
>
> *Paolo Aluffi*
> *Algebra: Chapter 0*

Group theory has a special place in abstract algebra: groups are simple and elegant, yet interesting enough to study – unlike monoids or magmas. The interesting thing about groups though is they transcend abstract algebra. What I mean by that is that in some sense groups are much more fundamental than what their simple algebraic structure may imply. Groups are algebraic incarnation of *actions* and *symmetries*.

Definition. If $\mathcal{C}$ is a category and a is an object of $\mathcal{C}$, an *action* of G in a is a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(a)$$

If ρ is injective ρ is called a faithful action.

Just like men – and women – a group is known by it's actions. Indeed, Caley's theorem establishes that every group is isomorphic to a subgroup of a permutation group. In other words, every group G is characterized by a set X and a faithful action of G in X, namely $X = G$ and

$$\begin{aligned} \rho : G &\longrightarrow S_X \\ g &\longmapsto \rho(g) : X \longrightarrow X \\ &\qquad\qquad\quad x \longmapsto g \cdot x \end{aligned}$$

This implies a group is essentially a group action in the category **Set** of sets. In the abstract *categorical* terms of the last definition, Caley's theorem amounts to the epigraph if this chapter – which is a fancy way of saying *a group is a thing that acts on some other* arbitrary *thing*.

We are usually interested in actions of a group G with some extra structure that *respect the extra structure of* G in some sense or another, such as (smooth) isometric actions of a Lie group over Riemannian manifolds. Notice that given a (connected) Riemannian manifold M, $\mathrm{Iso}(M)$ is a Lie group under composition – this is apparently called *the Myners-Steenrod theorem* – so saying *"*$\rho : G \longrightarrow \mathrm{Iso}(M)$ *is a smooth map"* actually makes sense.

Note. Clearly, every manifold is connected – otherwise we're really talking about multiple manifolds.

We'll look into some other examples of this when we discuss continuous representations of compact groups in chapter 2 and smooth representations of Lie groups, but before we get into that let's review some basic concepts in representation theory.

1.1 Representations

The single most well understood category in the entirety of mathematics is $\mathbb{C}$-**Vect**. Hence actions on $\mathbb{C}$-**Vect** are a natural starting point for understanding the behavior of a given group.

Definition 1.1.1. A complex vector space V equipped with an action $\rho : G \longrightarrow \mathrm{GL}(V)$ is called a *representation* of G – or alternatively a *linear action of* G. It's common to denote $\rho(g)$ by simply g when ρ can be inferred from the surrounding context.

Example 1.1.1. Given a group G, the group algebra $\mathbb{C}[G]$ equipped with the left multiplication

$$\begin{aligned} \rho : G \longrightarrow \quad & \mathrm{GL}(\mathbb{C}[G]) \\ g \longmapsto \rho(g) : \ & \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \\ & \qquad x \longmapsto g \cdot x \end{aligned}$$

is a representation of G.

For finite groups, $\mathbb{C}[G]$ is identified with the space $\mathbb{C}^G$ of functions $G \longrightarrow \mathbb{C}$, which is called *the regular representation of* G. Under this identification,

$$(g \cdot f)(h) = f(g^{-1}h)$$

Example 1.1.2. Given two representations V and W of G, $V \oplus W$ and $V \otimes W$ are both representations of G.

Example 1.1.3. Given a representation V of G and a representation W of H, the space $V \boxtimes W = V \otimes W$ with

$$(g, h)\ v \otimes w = gv \otimes hw$$

is a representation of $G \times H$.

Example 1.1.4. Given a representation V of G and a subgroup $H \subseteq G$, the space $\mathrm{Res}_H^G V = V$ is a representation of H – where the action of H is given by restricting $\rho : G \longrightarrow \mathrm{GL}(V)$ to H.

In particular, if V and W are representations of G then $\mathrm{Res}_{\Delta G \times G}^{G \times G} V \boxtimes W$ is identified with the representation $V \otimes W$ of G – where $\Delta G \times G$ is the subgroup $\{(g, g) : g \in G\} \subseteq G \times G$.

As you might have guessed, *representation theory* (of groups) is the study of *representations* of groups. The general goal of representation theory is to extract as much information about a given group G as possible from it's representations. It's also worth noting that understanding the relationship between representations is an integral part of representation theory. This brings us to the following definitions.

Definition 1.1.2. Given a group G and two representations V and W of G, a linear map

$$T : V \longrightarrow W$$

such that

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[T]{} & W \end{array}$$

for all $g \in G$ is called an *intertwing* operator or an *intertwiner*. Intertwing operators can be thought-of as *maps that preserve the action of* G.

Definition 1.1.3. Given a group G and a representation V of G, a subspace $W \subseteq V$ such that $GW \subseteq W$ is called a *subrepresentation* of V. If the V admits no proper subrepresentations – subrepresentations other than 0 and V – then V is called an *irreducible* representation.

This definitions naturally induce a category $\mathbf{Rep}(G)$, whose objects are representations of G and whose morphisms are intertwining operators. It turns out much of the structure of a group G can be reconstructed from $\mathbf{Rep}(G)$. In fact, a multitude of reconstruction theorems from the likes of Lev Pontryagin and Tadao Tannaka establish that, in certain contexts, the entirety of G can be reconstructed from the category $\mathbf{FinRep}(G)$ of finite-dimensional representations of G.

This hopefully establishes that *representation theory is useful*, but it also poses a problem. The reconstruction theorems require us to understand the whole of $\mathbf{Rep}(G)$ – or at least a large chunk of it. In other words, understanding individual representations won't get us anywhere, we need to study the collective behavior of *all* representations of group to be able to extract useful information from them.

Hence the classical problem in representation theory is classifying all representations of a given group G up to isomorphism. This turns out to be hard. However, the problem of classifying the finite-dimensional representations of a finite group G is a solved one. This will be the focus of our next section.

1.2 Finite-dimensional Representations of Finite Groups

The theory of finite-dimensional representations of finite groups usually serves as a first introduction to representation theory, while the simplicity and elegance of some of it's core arguments serve as inspiration for the theory used in more complicated settings.

The first instrumental piece of this theory is...

Lemma 1.2.1 (Schur)**.** *If V and W are irreducible representations of a group G and $T : V \longrightarrow W$ is an intertwining operator then T is either 0 or an isomorphism of representations.*

Proof. It suffices to note that $\ker T \subseteq V$ and $\operatorname{im} T \subseteq W$ are both subrepresentations. Hence T is either bijective or 0. ■

The key to solving the problem of the classification of finite-dimensional representations of finite groups lies in an innocent observation...

Theorem 1.2.1 (Maschke)**.** *Every finite-dimensional representation of a finite group is isomorphic to the direct sum of irreducible representations.*

We'll go over a proof of Maschke's theorem in the following chapter. For now, simply note that Maschke's theorem implies $\mathbf{FinRep}(G)$ is a semisimple category. In other words, understanding the irreducible representations of G is enough for reconstructing the entirety of $\mathbf{FinRep}(G)$. This observations, combined with something called *character theory* are enough to *completely annihilate* and *utterly destroy* our initial classification problem.

Unfortunately, not every group is finite. Our next question is...what happens if G is infinite? We begin our inquire by investigating the case of compact groups, which provides us a welcoming introduction to the world of topological groups and their representations. Many of the details omitted in this section will be covered in the next chapter. Please refer to [2], [6] and [4] for further details.

Chapter 2

Continuous Representations of Compact Groups

The following chapter is generally based on the third chapter of *A Journey Through Representation Theory: From Finite Groups to Quivers via Algebras* [2].

The algebraic theory of finite groups and their representations is quite rich, but it's infinite counterpart is generally lacking in comparison. Infinite groups are complex beasts on their own, and one usually has to endow them with geometric structure to get interesting results. The simplest geometric structure in town is topology, so one naturally pays special attention to topological groups – i.e. group objects in the category **Top** of topological spaces.

$$\begin{array}{ccc} \mathbf{GrpTop} & \longrightarrow & \mathbf{Grp} \\ \downarrow & & \downarrow \\ \mathbf{Top} & \longrightarrow & \mathbf{Set} \end{array}$$

Just as one only considers continuous group homomorphisms when dealing with topological groups, given a topological group G it's usual practice to ignore representations that *do not respect the topological structure of* G. But what is that supposed to mean?

Definition. A representation V of a topological group G is called *continuous* if V is a (Hausdorff) topological vector space and the map

$$(g, v) \longmapsto gv$$

is continuous.

An alternative formulation I've seen around is something along the lines of *a finite-dimensional representation* V *of* G *over* $\mathbb{R}$ *or* $\mathbb{C}$ *is continuous if*

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

is continuous. Note that $\mathrm{GL}(V)$ is a topological group, since $\mathrm{GL}(V)$ is one of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. The former definition is, of course, much more general, but I still find the latter more intuitive.

Likewise, we only consider continuous intertwining operators between continuous representations of G, and closed subrepresentations of continuous representations of G. Note that in this context, the phrase V *is irreducible* means *the only* closed *subrepresentations of* V *are* 0 *and* V.

I suppose this makes sense from the *categorical* perspective, but endowing groups and representations with a topological structure just for the sake of it isn't productive at all. Why is any of this useful?

Well, the point of all this is that the topology of G and it's representations allows us to reproduce many of the results of the theory of representations of finite groups, as we'll establish in the following. The parallels between representations of finite groups and representations of topological groups come mainly in the form of *tools derived from the Haar measure on compact groups* and the study of *unitary representations*. The closest thing in topology to finite groups are compact groups, so those will be the focus of the following chapter.

2.1 The Haar Measure on Compact Groups

As one would expect, the study of representations of finite groups is greatly facilitated by the fact that *finite groups are finite*. In practice this means things like

$$\sum_{g \in G} f(g)$$

are well defined for finite G. This is enormously helpful when building G-invariant construction on representations of G, since we can just take *any* construction on G and make it invariant by averaging over G.

The classic example of this is the existence of a G-invariant (positive definite) Hermitian inner product on a representation V of G over $\mathbb{C}$. Given *any* Hermitian product H on V, one can define

$$\langle v, u \rangle = \frac{1}{|G|} \sum_{g \in G} H(gv, gu)$$

and obtain a G-invariant Hermitian product on V. Now given a proper subrepresentation $W \subsetneq V$, notice $U = W^{\perp}$ is a subrepresentation of V. It then follows from the finite induction principle that Maschke's theorem holds. Ideally, we would like to extend some of this tools to the study of locally compact groups.

It turns out every Hausdorff locally compact group G admits a non-trivial G-invariant – either by left or right translations – Borel measure, and this measure is unique up to multiplication by a positive scalar. This measure is quite well-behaved too, in particular it is locally finite, regular and $\mu(U) > 0$ for every non-empty open subset $U \subseteq G$. The fact that G is required to be Hausdorff may seem like a huge limitation, but remember every T_0 topological group is actually Hausdorff.

This is called the – either *left* or *right* – *Haar measure* of G, and it allows us to reproduce some of the averaging arguments used for finite groups by replacing sums with integrals

$$\sum_{g \in G} f(g) \rightsquigarrow \int_G f(g) \, \mathrm{d}g$$

Note. From now on we'll denote the Haar measure of a Hausdorff locally compact group G by μ.

Example 2.1.1. Let G be a discrete group and μ be it's left Haar measure. Then $\{e\}$ is an open subset of G and therefore $\{e\}$ is measurable. Let $\lambda = \mu(\{e\})$.

Since $\{e\}$ is non-empty, $\lambda > 0$. Furthermore, since μ is left-invariant, $\mu(\{g\}) = \mu(\{e\}) = \lambda$ for each $g \in G$. Hence given a finite subset $X \subseteq G$

$$\mu(X) = \sum_{g \in X} \mu(\{g\}) = \sum_{g \in X} \lambda = \lambda \, |X|$$

If $X \subseteq G$ is a countably infinite subset then

$$\mu(X) = \sum_{g \in X} \mu(\{g\}) = \sum_{g \in X} \lambda = \infty$$

If $Y \subseteq G$ is uncountable, then there exists a countably infinite subset $X \subseteq Y$. Hence

$$\mu(Y) \geqslant \mu(X) = \infty$$

and therefore $\mu(Y) = \infty$. In conclusion, μ is a scalar multiple of the counting measure.

The left Haar measure of a locally compact group G isn't necessarily proportional to the right Haar measure of G, but this is the case for compact groups. Furthermore, if G is compact then G has finite measure and therefore the Haar measure of G can be normalized so that $\mu(G) = 1$. We won't normalize the Haar measure in this notes though, since we want to emphasize the parallels with finite-dimensional representations of finite groups.

The proof that every (Hausdorff) locally compact group admits a Haar measure is somewhat involved. A special class of locally compact groups are the Lie groups, also known as *groups that are also smooth manifolds* – i.e. group objects in the category **Diff** of smooth manifolds. It's much easier to show that every Lie group admits a Haar measure, precisely because Lie groups are orientable manifolds and therefore they admit a (G-invariant) volume form, as we'll establish in the following.

Note. As manifolds, Lie groups are locally Euclidean. Hence Lie groups are *locally locally compact*, i.e. locally compact.

2.1.1 Differential Forms

As previously mentioned, differential forms allow us to easily prove that every Lie group admits a Haar measure. We'll go over the definition of a differential form in a smooth manifold before proceeding to the proof mentioned above. The following comments are based on Marcos Alexandrino's notes on Riemannian Geometry [1].

Definition 2.1.1. Let M be an n-dimensional smooth manifold. Given $p \in G$, the tangent T_pM space at p is an n-dimensional real vector space. Now consider the real vector space $\wedge^k (T_pM)^*$ of k-linear functionals $\omega_p : T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R}$ such that

$$\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) \cdot \omega_p(v_1, \ldots, v_k)$$

for all $\sigma \in S_k$.

Since $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_i$ is a basis for T_pM,

$$\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : i_1 < \cdots < i_n\}$$

is a is a basis for $\wedge^k (T_pM)^*$ – where $\{\mathrm{d}x_i\}_i$ is the dual basis of $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_i$. This implies

$$\dim \wedge^k (T_pM)^* = \binom{n}{k}$$

This construction induces a vector bundle

$$\begin{array}{c} \coprod_{p \in M} \wedge^k (T_pM)^* \\ \big\downarrow \\ M \end{array}$$

commonly denoted by $\wedge^k T^*M$.

A (smooth) section of $\wedge^k T^*M$ – i.e. a smooth function $\omega : M \longrightarrow \wedge^k T^*M$ that takes each $p \in M$ to some $\omega_p \in \wedge^k (T_pM)^*$ – is called a *differential form of degree k*. Note that $\dim \wedge^n T_p^*M = 1$. Hence differential forms of degree n are called *differential forms of maximal degree*. Nowhere-vanishing – non-zero at every point – differential forms of maximal degree are called *volume forms*.

The set of all differential forms of degree k is usually denoted by $\Omega^k(M)$.

Notice $\Omega^k(M)$ is a $C^\infty(M)$-module where

$$(f \cdot \omega)_p = f(p) \cdot \omega_p$$

Moreover, an element $\omega \in \Omega^k(M)$ can be thought-of as a k-linear alternating functional

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M),$$

where $\mathfrak{X}(M)$ is the C^∞-module of (smooth) vector fields over M and

$$(\omega(V^1, \ldots, V^k))(p) = w_p(V^1_p, \ldots, V^k_p)$$

Why is this useful to us though? Well, besides various applications on differential geometry – which I'm completely unaware of – given a Lie group G, a differential form of maximal degree $\omega \in \Omega^n(G)$ can be integrated over of G. In other words,

$$\int_G \omega \in [-\infty, \infty]$$

is a thing that exists and we can use this to build a measure over G. I'll spear the reader from the boring details of the construction of the integral of a differential form, all we need to know at the moment is that integration of volume forms is something generally well behaved. Please refer to [1] for more information on this topic.

Theorem 2.1.1. *Every Lie group G admits a left Haar measure.*

Proof. Suppose $\dim G = n$. Given $g \in G$, the left translations $\ell_{g^{-1}} : G \longrightarrow G$ by g^{-1} is a diffeomorphism that takes g to e. Consider the induced isomorphism

$$\ell^*_{g^{-1}} : \wedge^n T^*_e G \longrightarrow \wedge^n T^*_g G$$

given by $\wedge^n (d\ell_{g^{-1}})^*_g$ – where $(d\ell_{g^{-1}})^*_g : T^*_e G \longrightarrow T^*_g G$ is the dual of $(d\ell_{g^{-1}})_g : T_g G \longrightarrow T_e G$.

Let $\omega_e = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \wedge^n T^*_e G$. Then the map

$$\begin{aligned} \omega : G &\longrightarrow \wedge^n T^* G \\ g &\longmapsto \ell^*_{g^{-1}}(\omega_e) \end{aligned}$$

is a volume form. We claim ω is left-invariant too. Indeed,

$$\begin{aligned} (\ell^*_g \omega)_h &= \wedge^n (d\ell_g)^*_h \, \omega_{gh} \\ &= \wedge^n (d\ell_g)^*_h \wedge^n (d\ell_{h^{-1}g^{-1}})^*_{gh} \, \omega_e \\ &= \wedge^n ((d\ell_g)^*_h (d\ell_{h^{-1}g^{-1}})^*_{gh}) \, \omega_e \\ &= \wedge^n ((d\ell_{h^{-1}g^{-1}})_{gh} (d\ell_g)_h)^* \, \omega_e \\ \text{(chain rule to the rescue)} &= \wedge^n (d\ell_{h^{-1}})^*_h \, \omega_e \\ &= \ell^*_{h^{-1}}(\omega_e) \\ &= \omega_h \\ \therefore \ell^*_g \omega &= \omega \end{aligned}$$

Since $\{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\}$ is basis of the $C^\infty(G)$-module $\Omega^n(G)$, there exists $f \in C^\infty(G)$ such that $\omega = f \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Since ω is nowhere-vanishing, $f(g) \neq 0$ for all $g \in G$. Notice $f(e) = 1$. This implies $f(g) > 0$ for all $g \in G$.

Hence

$$\mu(K) = \int_K \omega \geqslant 0$$

for each compact subset $K \subseteq G$.

Clearly, $\mu(K) < \infty$, and it's easy to check that $\mu : \mathcal{K}(G) \longrightarrow [0, \infty)$ is a countably additive monotonic function too. Moreover,

$$\mu(gK) = \int_{\ell_g(K)} \omega = \int_K \ell_g^* \omega = \int_K \omega = \mu(K)$$

and therefore $\mu : \mathcal{K}(G) \longrightarrow [0, \infty)$ is a left-invariant pre-measure.

By Caratheodory's extension theorem, μ can be extended to every Borel subset of G, yielding a left-invariant Borel measure $\mu : \mathfrak{B}(G) \longrightarrow [0, \infty]$ – see San Martin's construction of the Haar measure [5] for further details. Furthermore, since ω is nowhere-vanishing, μ is non-trivial. We are done. ■

Note. Notice we've also proved G is orientable – i.e. G admits a volume form.

The proof that every Lie group admits a right Haar measure is essentially the same – just replace ℓ with $\imath$ everywhere. Let's assume that every locally compact group admits a Haar measure and proceed to some applications of this fact.

2.2 Unitary Representations

Unitary representations play a special role in the representation theory of compact groups. In particular, there's a formulation o Schur's lemma that holds for unitary representations, which allows us to reproduce many of the results of the theory of representations of finite groups in the context of compact groups.

As one would expect, a unitary representation of a topological group G is a representation in which every element $g \in G$ acts *unitarily* – i.e. g is a unitary operator. In other words…

Definition 2.2.1. A continuous representation V of a topological group G is called *unitary* if V is a Hilbert space and

$$\langle v, w \rangle = \langle gv, gw \rangle$$

for all $g \in G$.

Example 2.2.1. If G is a compact group then the space $L^2(G)$ of functions $f : G \longrightarrow \mathbb{C}$ such that

$$\int_G |f(g)|^2 \, \mathrm{d}g$$

exists is a unitary representation of G, where the action of $g \in G$ is given by

$$(g \cdot f)(h) = f(g^{-1}h)$$

and

$$\langle f_1, f_2 \rangle = \frac{1}{\mu(G)} \int_G f_1(g)\overline{f_2(g)} \, \mathrm{d}g$$

This is called the regular representation of G.

When G is finite and discrete the integral over G is just the regular summation and $L^2(G)$ is the space of all complex-valued functions on G – also known as $\mathbb{C}[G]$. In other words, this is a strict generalization of the regular representation of finite groups.

I'll restate some of the results presented in *A Journey Through Representation Theory: From Finite Groups to Quivers via Algebras* [2] in here for the sake completion and to highlight some of the parallels between representations of finite groups and unitary representations of compact groups. For the missing proofs see [2].

As we'll establish in the following, unitary representations are in many regards the compact analogous of finite-dimensional representations of finite groups.

Lemma 2.2.1. *Every non-zero unitary representation V of a compact group G has a non-zero finite-dimensional (closed) subrepresentations.*

Note. This proof is quite involved so buckle your seat-belts.

Proof. Given $u \in V$ with $\|u\| = 1$, definite

$$\begin{aligned} Q : V &\longrightarrow V \\ v &\longmapsto \int_G gTg^{-1}v \, \mathrm{d}g \end{aligned}$$

where

$$\begin{aligned} T : V &\longrightarrow V \\ v &\longmapsto \langle v, u \rangle u \end{aligned}$$

We want to establish that $\ker Q - \lambda \, \mathrm{Id}$ is a non-zero finite-dimensional subrepresentation of V for some $\lambda > 0$.

Notice Q is a self-adjoint intertwining operator. Furthermore

$$\begin{aligned} \langle v, Qv \rangle &= \left\langle v, \int_G \langle g^{-1}v, u \rangle gu \, \mathrm{d}g \right\rangle \\ &= \int_G \langle v, gu \rangle \overline{\langle v, gu \rangle} \, \mathrm{d}g \\ &= \int_G |\langle v, gu \rangle|^2 \, \mathrm{d}g \\ &\geqslant 0 \end{aligned}$$

so Q is semipositive. We claim Q is compact too.

Indeed, it follows from the Cauchy-Schwartz inequality that T is bounded. Moreover,

$$\dim \operatorname{im} T = \dim \langle u \rangle = 1$$

so gTg^{-1} is a bounded operator with 1-dimensional image for all $g \in G$. This implies

$$Q = \int_G gTg^{-1} \, \mathrm{d}g$$

is compact – since the ideal $\mathcal{C}(V) \subseteq \mathcal{B}(V)$ of all compact operators $V \longrightarrow V$ is the closure of the ideal $\mathcal{F}(V) \subseteq \mathcal{B}(V)$ of bounded operators $V \longrightarrow V$ with finite-dimensional image.

In other words, Q is a compact semipositive self-adjoint intertwining operator. Hence

$$\lambda = \sup_{\|v\|=1} \langle v, Qv \rangle$$

is either 0 or an eigenvalue of T. Notice

$$\langle u, Qu \rangle = \int_G |\langle u, gu \rangle|^2 \, \mathrm{d}g > 0$$

since

$$g \longmapsto |\langle u, gu \rangle|^2$$

is a continuous map that takes every element of G to a non-negative real number and

$$|\langle u, u \rangle|^2 = 1 > 0$$

This implies $\lambda > 0$, so λ is a positive eigenvalue of Q. Let $W = \ker Q - \lambda \, \mathrm{Id}$. Notice W is a (closed) subrepresentation of V, since it is the kernel of a continuous intertwining operator. Furthermore, W is non-zero since it is the eigenspace of T associated to $\lambda > 0$.

In conclusion, since Q is compact and self-adjoint and $\lambda \neq 0$, it follows from the spectral theorem that W is finite-dimensional. Finally, W is a finite-dimensional (closed) subrepresentation of V! ■

Corollary 2.2.1. *Every irreducible unitary representation of a compact group is finite-dimensional.*

Corollary 2.2.2 (Schur)**.** *If V as an irreducible unitary representation of a compact group G and $T : V \longrightarrow V$ is a continuous intertwining operator, then there exists $\lambda \in \mathbb{C}$ such that $T = \lambda \operatorname{Id}$.*

Proof. Since V is unitary, V is finite-dimensional. It then follows that T has at least one eigenvalue, so $\ker T - \lambda \operatorname{Id} \neq 0$ for some $\lambda \in \mathbb{C}$.

Notice $T - \lambda \operatorname{Id}$ is a continuous intertwiner. This implies that $\ker T - \lambda \operatorname{Id}$ is a (closed) subrepresentation of V. Since $\ker T - \lambda \operatorname{Id} \neq 0$, $\ker T - \lambda \operatorname{Id} = V$ and therefore $T = \lambda \operatorname{Id}$. ■

Corollary 2.2.3. *Let V and W be two irreducible unitary representations of a compact group G and $T : V \longrightarrow W$ be a continuous intertwining operator. Then either $T = 0$ or there exists $\lambda > 0$ such that λT is an isometry.*

Proof. Suppose $T \neq 0$. It follows from Schur's lemma that

$$T^* T = \mu \operatorname{Id}$$

for some $\mu \in \mathbb{C}$. We want to establish that μ is a positive real number.

Since μ is an eigenvalue of the bounded semipositive self-adjoint operator T^*T, μ is a non-negative real number. Suppose $\mu = 0$. Then $T^*T = 0$ and therefore T is a not invertible. ↯

Hence $\mu \neq 0$ and therefore $\mu > 0$. Let $\lambda = \frac{1}{\sqrt{\mu}}$ and $U = \lambda T$. Then

$$\begin{aligned} U^* U &= \frac{1}{\sqrt{\mu}} T^* \frac{1}{\sqrt{\mu}} T \\ &= \frac{1}{|\mu|} \, \mu \operatorname{Id} \\ &= \operatorname{Id} \end{aligned}$$

In other words, $U = \lambda T$ is an intertwining isometry. ■

Corollary 2.2.4. *Every irreducible unitary representation of a compact Abelian group is 1-dimensional.*

Proof. It suffices to note that $g : V \longrightarrow V$ is a continuous intertwiner for each $g \in G$, precisely because G is Abelian. ■

Theorem 2.2.1. *Every irreducible continuous representation of a compact group G is isomorphic to a subrepresentation of the regular representation $L^2(G)$ of G.*

Corollary 2.2.5. *Every irreducible continuous representation of a compact group G is isomorphic to a unitary representation.*

Proof. It suffices to note that $L^2(G)$ is a unitary representation. ■

It turns out Schur's lemma, corollary 2.2.4 and corollary 2.2.5 all hold for locally compact groups too. This is essential for establishing the connection between Pontryagin's duality and representation theory. For more information on the topic please refer to *Fourier Analysis on Number Fields* [3].

Corollary 2.2.6. *Every irreducible continuous representation of a compact group G is finite-dimensional.*

Theorem 2.2.2. *Every unitary representation of a compact group G is completely reducible. In other words, given a unitary representation V of G and a (closed) subrepresentation $W \subseteq V$, there exists a (closed) subrepresentation $U \subseteq V$ such that*

$$V \cong W \oplus U$$

Proof. Consider $U = W^\perp$. Note U is a closed subspace of V. Furthermore, given $u \in U$, $w \in W$ and $g \in G$, it follows from the fact that $g^{-1}w \in W$ that

$$\langle gu, w\rangle = \langle u, g^{-1}w\rangle = 0$$

This implies U is G-invariant – i.e. a subrepresentation of V. In conclusion, U is a (closed) subrepresentation of V and

$$V \cong W \oplus U$$

■

Corollary 2.2.7 (Maschke). *Every finite-dimensional continuous representation of a compact group G is semisimple.*

Note. The proof of Maschke's theorem for compact groups is precisely the same as the proof for finite groups!

Proof. Let V be a finite-dimensional continuous representation of G. Given a Hermitian inner product H, consider

$$\langle v, u\rangle = \frac{1}{\mu(G)} \int_G H(gv, gh)\, \mathrm{d}g$$

Our goal is to show that V is a unitary representation under $\langle\,,\rangle$.

Note $\langle\,,\rangle$ is a G-invariant inner product on V. We want to establish that the topology of V is the topology induced by $\langle\,,\rangle$.

Since $\dim V < \infty$, V is isomorphic to $\mathbb{C}^n$ as a topological vector space – where $n = \dim V$. Hence any two metrics on V derived from norms are equivalent. In particular, the metric derived from $\langle\,,\rangle$ and the Euclidean distance are equivalent.

This implies the topology of V is the topology induced by $\langle\,,\rangle$. Furthermore, since the Euclidean space is complete under the Euclidean distance, V is complete under the distance induced by $\langle\,,\rangle$.

So V is a Hilbert space under $\langle\,,\rangle$ and $\langle\,,\rangle$ is G-invariant. In other words, V is a unitary representation of G. Hence V is completely reducible. In conclusion, since V is finite-dimensional and completely reducible, V is semisimple. ■

Note that for finite G, μ is simply the counting measure and the integral is the standard summation, so this construction is compatible with the construction used for finite groups.

2.3 The Peter-Weyl Theorem & Character Theory

Another important aspect of the theory of representations of compact groups is something called *character theory*. Given a finite-dimensional representation V of a compact group G, consider the function

$$\begin{aligned} \chi_V : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \mathrm{Tr}(g\restriction_V) \end{aligned}$$

This is called the *character of V*. If V is irreducible, χ_V is called an irreducible character. Clearly, the characters of two isomorphic representations coincide, since the trace of a matrix is invariant under changes of basis. Hence characters are invariants of representations. The same argument can be used to show that characters of representations of a group G are constant in the conjugacy classes of G – i.e. characters are *class functions*. Moreover, it's easy to check that $\chi_{V\oplus W} = \chi_V + \chi_W$.

Why are we doing all this though? Well, it turns out characters of finite groups are not only invariants, they are *perfect invariants*. In other words, a finite-dimensional representation of a finite group is completely determined by it's character – i.e. if V and W are finite-dimensional representations of a finite group G and $\chi_V = \chi_W$ then $V \cong W$. This is a remarkably powerful result, which follows immediately from the fact that…

Theorem 2.3.1. *The characters of irreducible representations of a finite group G are orthonormal in $L^2(G)$.*

Hence...

Corollary 2.3.1. *A finite-dimensional representation of a finite group G is completely determined by it's character. In other words, the ring $R(G)$ of virtual representations of G is a free $\mathbb{Z}$-module.*

Proof. Given a finite-dimensional representation V of G, it follows from Maschke's theorem that

$$V \cong \bigoplus_{i=1}^{n} V_i$$

for some irreducible representations $V_1, V_2, \ldots V_n$ of G. This implies $\chi_V = \chi_{V_1} + \cdots + \chi_{V_n}$ is a linear combination of the irreducible characters of G.

Since the irreducible characters of G are orthonormal, they for a basis of the subspace of functions $G \longrightarrow \mathbb{C}$ spanned by the irreducible characters themselves. This implies that χ_V can be uniquely expressed as a linear combination of irreducible characters of G.

Now let W be a finite-dimensional representation of G such that $\chi_W = \chi_V$. Suppose

$$W \cong \bigoplus_{i=1}^{m} W_i$$

for some irreducible representations $W_1, W_2, \ldots, W_m$ of G. It then follow that

$$\chi_{V_1} + \cdots + \chi_{V_n} = \chi_V = \chi_W = \chi_{W_1} + \cdots + \chi_{W_m},$$

so $W_i = V_i$. This establishes that $V \cong W$. ■

This comes in handy for verifying that a representation is irreducible, since...

Corollary 2.3.2. *A finite-dimensional representation V of G is irreducible if and only if*

$$\langle \chi_V, \chi_V \rangle = 1$$

Corollary 2.3.2 can be used to show that...

Theorem 2.3.2. *If V is the set of all irreducible representations of G up to isomorphism then*

$$\mathbb{C}[G] \cong \bigoplus_{V \in \widehat{G}} \dim V \cdot V$$

Another important consequence of theorem 2.3.1 is...

Theorem 2.3.3. *The irreducible characters of a finite group G form an orthonormal basis of the space $\mathcal{C}(G)$ of class functions $G \longrightarrow \mathbb{C}$.*

For a proof of theorem 2.3.3 see [6]. This implies that the number of irreducible representations of a finite group G – up to isomorphism – is precisely the number of conjugacy classes of G. The goal of this section is to generalize this results to compact groups. The case of compact groups is more involved, since we lack some of the tools used for finite groups. Nevertheless, this results can be generalized by using something called *matrix coefficients.*

Definition 2.3.1. Given a unitary representation V of a compact group G and $v, w \in V$. Consider

$$\begin{aligned} f_{v,w} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle gv, w \rangle \end{aligned}$$

This is called a *matrix coefficient* of V.

This is particularly useful for computing the character of finite-dimensional representations, since if $g = (a_{ij})_{ij}$ in an orthonormal basis $\{e_1, \ldots, e_n\}$ then $f_{e_i,e_j}(g) = a_{ij}$. Hence

$$\chi_V(g) = \mathrm{Tr}(g \restriction_V) = \sum_{i=1}^{n} f_{e_i,e_i}(g)$$

Using matrix coefficients, it's easy enough to show that theorem 2.3.1, corollary 2.3.1 and corollary 2.3.2 all hold for compact groups [2], but ideally we would also like to show that theorem 2.3.2 holds. Moreover, we would like to show that the irreducible characters of a compact G span the subspace $\mathcal{C}(G) \subseteq L^2(G)$ of class functions – the word *span* is doing some legwork here, we'll get to the precise formulation of this. Proving this requires *the Peter-Weyl theorem*. This theorem may not seem particularly interesting on it's own, but it is essential for generalizing theorem 2.3.2 and theorem 2.3.3.

Theorem 2.3.4 (Peter-Weyl)**.** *The matrix coefficients of all irreducible representations of compact group G span a dense subspace in $L^2(G)$.*

Lemma 2.3.1. *The map*

$$\begin{aligned} \Phi : \widehat{\bigoplus_{V \in \widehat{G}}} V^* \boxtimes V &\longrightarrow L^2(G) \\ e_i^* \otimes e_j &\longmapsto f_{e_j,e_i} \end{aligned}$$

is an isomorphism of $G \times G$-representations where the action of $G \times G$ in $L^2(G)$ is given by

$$((g,h)f)(k) = f(g^{-1}kh)$$

Proof. It's clear that Φ is linear, and it follows from the Peter-Weyl theorem that Φ is also surjective. To see that Φ is an intertwiner, it suffices to observe that

$$\begin{aligned} ((g,h)\phi(e_i^* \otimes e_j))(k) &= ((g,h)f_{e_j,e_i})(k) \\ &= f_{e_j,e_i}(g^{-1}kh) \\ &= \langle g^{-1}khe_j, e_i \rangle \\ &= \langle khe_j, ge_i \rangle \\ &= f_{he_j,ge_i}(k) \\ &= (\phi(ge_i^* \otimes he_j))(k) \\ &= (\phi((g,h)e_i^* \otimes e_j))(k) \\ &= (g,h)\phi(v \otimes w) = \phi((g,h)v \otimes w) \end{aligned}$$

Now given $V \in \widehat{G}$, notice $\|\chi_{V^* \boxtimes V}\| = \|\chi_{V^*}\| \cdot \|\chi_V\| = 1$. It then follow from corollary 2.3.2 that $V^* \boxtimes V$ is irreducible. Hence $\ker \Phi \restriction_V$ is either 0 or V. Since $\Phi(e_1^* \otimes e_1) = f_{e_1,e_1} \neq 0$, $\ker \Phi \restriction_V = 0$. This implies Φ is injective.

In conclusion, Φ is an isomorphism of representations. ■

Again, this results aren't that appealing on their own. What we're really interested in is...

Corollary 2.3.3 (Peter-Weyl)**.**

$$L^2(G) \cong \widehat{\bigoplus_{V \in \widehat{G}}} \dim V \cdot V$$

Proof. Notice $G \times \{e\} \cong G$ as topological groups. On the one hand, $\operatorname{Res}^{G\times G}_{G\times\{e\}} L^2(G)$ is precisely the regular representation $L^2(G)$. On the other hand, given $V \in \widehat{G}$ it's easy to check that

$$\begin{aligned}
\operatorname{Res}^{G\times G}_{G\times\{e\}} V^* \boxtimes V &\cong V^* \otimes \bigoplus_{i=1}^{\dim V} \mathbb{C} \\
&\cong \bigoplus_{i=1}^{\dim V} V^* \otimes \mathbb{C} \\
&\cong \bigoplus_{i=1}^{\dim V^*} V^* \\
&= \dim V^* \cdot V^*
\end{aligned}$$

Moreover, since V is irreducible, V^* is irreducible. This implies

$$\operatorname{Res}^{G\times G}_{G\times\{e\}} \widehat{\bigoplus_{V\in\widehat{G}}} V^* \boxtimes V \cong \widehat{\bigoplus_{V\in\widehat{G}}} \dim V \cdot V$$

Hence Φ can be thought of as an isomorphism of G-representations that takes $\widehat{\bigoplus}_{V\in\widehat{G}} \dim V \cdot V$ to $L^2(G)$. ■

Corollary 2.3.4 (Peter-Weyl)**.** *The irreducible characters of G form an orthonormal basis of the subspace $\mathcal{C}(G)$ of class-functions – in the sense that they span a dense subspace in $\mathcal{C}(G)$.*

Note. Yet another lengthy proof lies ahead.

Proof. Given an irreducible representation V of G consider the algebra $\operatorname{End}(V)$ with

$$T \cdot S = \frac{1}{\dim V} TS$$

and

$$\begin{aligned}
\Psi : \widehat{\bigoplus_{V\in\widehat{G}}} \operatorname{End}(V) &\longrightarrow L^2(G) \\
E_{ij} &\longmapsto f_{e_j, e_i}
\end{aligned}$$

We want to establish that Ψ is an isomorphism of algebras between $\widehat{\bigoplus}_{V\in\widehat{G}} \operatorname{End}(V)$ and $L^2(G)$ equipped with the convolution product

$$(f_1 * f_2)(g) = \frac{1}{\mu(G)} \int_G f_1(h) f_2(h^{-1}g)\, \mathrm{d}h,$$

This may all seem *extremely arbitrary* – and indeed it is! The point of all this is we'll show that the center of $L^2(G)$ is $\mathcal{C}(G)$ and the center of $\widehat{\bigoplus}_{V\in\widehat{G}} \operatorname{End}(V)$ is precisely what we want it to be. First of all, note that Ψ factors through Φ.

$$\begin{array}{ccc}
\widehat{\bigoplus}_{V\in\widehat{G}} V^* \boxtimes V & \xrightarrow{\sim} & \widehat{\bigoplus}_{V\in\widehat{G}} \operatorname{End}(V) \\
& \underset{\Phi}{\searrow} & \downarrow \Psi \\
& & L^2(G)
\end{array}$$

This implies Ψ is a linear isomorphism. To see that Φ is a homomorphism of algebras, it suffices to observe that

$$\begin{aligned}
(\Psi(E_{ij}\cdot E_{jk}))(g) &= \frac{1}{\dim V}(\Psi(E_{ik}))(g) \\
&= \frac{1}{\dim V} f_{e_k,e_i}(g) \\
&= \frac{1}{\dim V}\langle e_j, e_j\rangle\langle ge_k, e_i\rangle \\
\text{(see theorem 2.1 of [2])} &= \langle f_{e_j,e_i}, f_{e_j,ge_k}\rangle \\
&= \frac{1}{\mu(G)}\int_G \langle he_j, e_i\rangle\overline{\langle he_j, ge_k\rangle}\,\mathrm{d}h \\
&= \frac{1}{\mu(G)}\int_G \langle he_j, e_i\rangle\langle h^{-1}ge_k, e_j\rangle\,\mathrm{d}h \\
&= \frac{1}{\mu(G)}\int_G f_{e_j,e_i}(h) f_{e_k,e_j}(h^{-1}g)\,\mathrm{d}h \\
&= (f_{e_j,e_i} * f_{e_k,e_j})(g) \\
&= (\Psi(E_{ij}) * \Psi(E_{jk}))(g) \\
&\therefore \Psi(T\cdot S) = \Psi(T) * \Psi(S)
\end{aligned}$$

Notice that $\mathcal{C}(G)$ is the center of $L^2(G)$ under the convolution product. Moreover, the center of $\mathrm{End}(V)$ is clearly $\mathbb{C}\,\mathrm{Id}$. We claim that the image of $\mathbb{C}\,\mathrm{Id}$ under Ψ is $\mathbb{C}\chi_V$. Indeed,

$$(\Psi(\mathrm{Id}))(g) = \sum_{i=1}^{n}(\Psi(E_{ii}))(g) = \sum_{i=1}^{n} f_{e_i,e_i}(g) = \chi_V(g)$$

Hence

$$\begin{aligned}
\mathcal{C}(G) &= Z(L^2(G)) \\
&= \Psi\left(Z\left(\widehat{\bigoplus_{V\in\widehat{G}}}\mathrm{End}(V)\right)\right) \\
&= \widehat{\bigoplus_{V\in\widehat{G}}} Z(\Psi(\mathrm{End}(V))) \\
&= \widehat{\bigoplus_{V\in\widehat{G}}} \mathbb{C}\chi_V
\end{aligned}$$

We are done. ■

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